THE ASYMPTOTIC LINES ON THE ANCHOR RING

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THE purpose of this paper is to determine the equations of the asymptotic lines on the surface of the anchor ring when its equations are expressed in parameter form and, with the aid of elliptic functions, to examine their geometrical properties.* The first three sections are devoted to the equations of asymptotic lines in general, to their applications to the anchor ring, and to the expression of the equations of the asymptotic lines in terms of the elliptic functions of Jacoby. After determining the general form of the asymptotic curves, the anchor rings for which they are closed will be discussed. The paper concludes with the computation of a numerical example.

1. Asymptotic Lines in General. Consider a surface whose equations are

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

and a plane tangent to that surface at a point M whose coordinates are (u, v). The distance of any point $(u + h, v + k)$ on the surface in the neighborhood of M from this tangent plane is represented in Bianchi's notation† by

$$\delta = \tfrac{1}{2}(Dh^2 + 2D'hk + D''k^2) + \eta, \tag{1}$$

* The only literature I have found on this subject is a brief summary of an article by T. Matoda, "Asymptotic lines on a circular ring," *Tokio Math. Soc.*, vol. 4, pp. 217–219, in the *Jahrbuch über die Fortschritte der Mathematik*, vol. 21, p. 765, and also an article by Prof. Maschke, "Asymptotic lines on a circular ring," *Bull. Amer. Math. Soc.*, vol. 2, 1895, pp. 19–21, in which the equations of the lines are briefly reduced to a form involving Weierstrassian elliptic functions without any corresponding geometrical investigation.

† L. Bianchi, *Vorlesungen über Differentialgeometrie*, translation by M. Lukat (1896), pp. 61, 87, 104.

where η is an infinitesimal of the third order in h and k. The expressions for D, D' and D'' are:

$$(2)\qquad D = \frac{1}{\sqrt{EG-F^2}}\begin{vmatrix} x_{uu} & y_{uu} & z_{uu} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix},$$

$$D' = \frac{1}{\sqrt{EG-F^2}}\begin{vmatrix} x_{uv} & y_{uv} & z_{uv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix},$$

$$D'' = \frac{1}{\sqrt{EG-F^2}}\begin{vmatrix} x_{vv} & y_{vv} & z_{vv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix},$$

where x_u, x_v are the partial derivatives of x with respect to u and v, and as usual

$$(3)\qquad E = x_u^2 + y_u^2 + z_u^2, \quad F = x_u x_v + y_u y_v + z_u z_v, \quad G = x_v^2 + y_v^2 + z_v^2.$$

The surface in the neighborhood of M lies on one side of, or is cut by, the tangent plane according as δ retains or changes its sign. This δ will always be positive for sufficiently small values of h and k if the two factors of $Dh^2 + 2D'hk + D''k^2$ are conjugate imaginary factors, i. e., if $DD'' - D'^2 > 0$; in this case the surface is said to have elliptic curvature at the point M. If $DD'' - D'^2 < 0$, δ may take positive or negative values for small values of h and k and the curvature at M is said to be hyperbolic. Parabolic curvature occurs at points where $DD'' - D'^2 = 0$.

The curve in the uv-plane corresponding to the equation

$$(4)\qquad \delta = \tfrac{1}{2}(Dh^2 + 2D'hk + D''k^2),$$

where δ is constant and η has been neglected, is approximately the image in the uv-plane of the section of the surface by a plane parallel to, and at a distance δ from the tangent plane.

Two systems of curves on a surface are conjugate systems if the directions of their tangents at every point on the surface correspond to directions in the uv-plane which are conjugate with respect to the conic (4). From the theory of conic sections, the condition for this is*

* See also: Bianchi, *Vorlesungen über Differentialgeometrie*, p. 108.

(5) $$D\,du\,\delta u + D'(du\,\delta v + dv\,\delta u) + D''dv\,\delta v = 0,$$

where d and δ are used to distinguish between the increments along the two curves. In particular, *if a curve on the surface is such that the direction of the tangent at any point is conjugate to itself, the curve is called an asymptotic line on the surface.* It follows immediately from the last equation that the differential equation for the asymptotic lines is

$$D\,du^2 + 2D'du\,dv + D''dv^2 = 0.$$

As this is a quadratic in du and dv, the asymptotic lines evidently form a double system. If they are real, the equation must have real roots. It follows therefore that the real asymptotic lines lie only on that part of the surface where $DD'' - D'^2 < 0$.

To summarize: *The asymptotic lines on the surface $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ form a double system of curves whose differential equation is*

(6) $$D\,du^2 + 2D'\,du\,dv + D''\,dv^2 = 0,$$

D, D' and D'' having the values given in equations (2). *They are real only on that part of the surface where $DD'' - D'^2 < 0$.*

2. The Equations of the Asymptotic Lines on the Anchor Ring. Let us now apply the general formulas just obtained to the particular case of the anchor ring. The anchor ring to be considered is generated by the revolution of a circle with radius b and center at $(a, 0, 0)$ about the

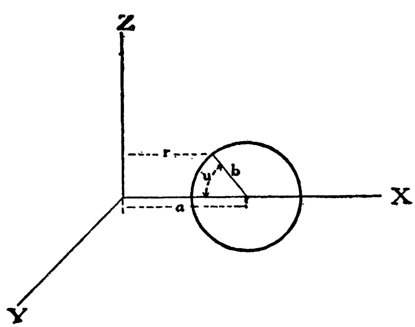

FIG. 1.

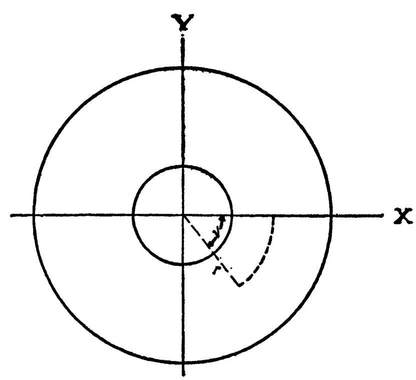

FIG. 2.

z-axis. Instead of using the equation in the form $F(x, y, z) = 0$, we may express x, y and z as functions of two parameters r and v or of u and v, whose meaning is indicated in the accompanying figures.

By referring to these figures, it is easy to see that the equations of the anchor ring in terms of u, v are:

$$x = (a - b\cos u)\cos v,\qquad (7)$$
$$y = (a - b\cos u)\sin v,$$
$$z = b\sin u,$$

the parameter lines being the meridians and parallels on the surface.

By means of the formulas (2) and (3), E, F, G, D, D' and D'' may be calculated, with the following results ;—

$$E = b^2,\ F = 0,\ G = (a - b\cos u)^2,\ \sqrt{EG - F^2} = b(a - b\cos u),$$
$$D = -b,\ D' = 0,\qquad D'' = (a - b\cos u)\cos u.$$

It is evident that F must equal zero, since x_u, y_u, z_u and x_v, y_v, z_v are the direction cosines of the parameter lines, which intersect orthogonally.

Substituting these values of D, D', and D'' in the equation (6) of the asymptotic lines, it becomes

$$-b\,du^2 + (a - b\cos u)\cos u\,dv^2 = 0,\qquad (8)$$

from which we derive

$$dv = \frac{\pm\sqrt{b}}{\sqrt{\cos u(a - b\cos u)}}\,du,\qquad (9)$$

or

$$v = \pm\sqrt{b}\int_0^u \frac{du}{\sqrt{\cos u\,(a - b\cos u)}} + C.\qquad (10)$$

The constant C may be determined so that the asymptotic line passes through an arbitrary point (u_0, v_0) on the ring, provided (u_0, v_0) lies in the portion of the surface where $DD'' - D'^2 < 0$. Then

$$C = v_0 \mp \sqrt{b}\int_0^{u_0} \frac{du}{\sqrt{\cos u(a - b\cos u)}},\qquad (11)$$

and

$$v = \pm\sqrt{b}\int_0^u \frac{du}{\sqrt{\cos u(a - b\cos u)}} + v_0 \mp \sqrt{b}\int_0^{u_0} \frac{du}{\sqrt{\cos u(a - b\cos u)}}.\qquad (12)$$

By a rotation about the z-axis, the equation for any asymptotic line can therefore be finally reduced to one of the forms

$$(13) \qquad v = \pm \sqrt{b} \int_0^u \frac{du}{\sqrt{\cos u (a - b \cos u)}}.$$

In §1 we found that these lines can be real only when $DD'' - D'^2 = - b \cos u \, (a - b \cos u) < 0$. Since $a > b$, we have $a - b \cos u > 0$ and therefore this inequality holds only when $-\frac{\pi}{2} < u < \frac{\pi}{2}$. These are exactly the values for which the denominator in the integral of (13) is real.

Our results are then as follows:

1) *The equations of any asymptotic lines on the anchor ring may, by a rotation about the z-axis, be reduced to one of the forms* (13).

2) *The asymptotic lines are real only on the inner half of the ring, that is, in the region* $-\frac{\pi}{2} < u < \frac{\pi}{2}$.

3. The Equations in terms of Elliptic Functions. The purpose of the present section is to express the equations of the asymptotic lines in terms of the elliptic functions. That the integral (13) is really an elliptic integral, appears when u is expressed in terms of $r = a - b \cos u$ (see figure 1). Since $\cos u = (a - r)/b$, the integral (13) becomes

$$(14) \qquad v = \pm b \int_{a-b}^{r} \frac{dr}{\sqrt{r(a-r)[(a-r)^2 - b^2]}}.$$

Hence r, and therefore also u, are elliptic functions of v.

To reduce this integral to the normal form, Durège's method* will be used. The integral to be reduced is in the form $\int_{r_1}^{r_2} \frac{dr}{\sqrt{AR^2}}$, where $A = 1$ and the four roots of $R^2 = 0$ are 0, $a - b$, a, and $a + b$. We will now introduce a new variable ξ, so chosen that the values of ξ which correspond to 0, $a - b$, a, and $a + b$ are $-1/k$, -1, 1, and $1/k$ respectively. If the four roots of the

* H. Durège, *Theorie der elliptischen Functionen* (1887), p. 34.

equation in r are designated by π, p, q, and κ, we have the following correspondence between the values of ξ and r.

$$\left\{\begin{array}{lllll}\xi = & 1/k, & 1, & -1, & -1/k, \\ r = & \pi, & p, & q, & \kappa,\end{array}\right. \tag{15}$$

where $\pi = a + b$, $p = a$, $q = a - b$, $\kappa = 0$.

The following formulas are the ones given by Durège to effect the transformation to the normal form when the roots of $R^2 = 0$ are all real. The relation between r and ξ is given by*

$$\frac{r-p}{r-q} = P\frac{1-\xi}{1+\xi}, \tag{16}$$

where

$$P = \pm\sqrt{\frac{(p-\pi)(p-\kappa)}{(q-\pi)(q-\kappa)}}. \tag{17}$$

The value of k is determined from

$$\frac{1-k}{1+k} = +\sqrt{\frac{(p-\pi)(q-\kappa)}{(q-\pi)(p-\kappa)}}. \tag{18}$$

The values of the old and new integrals are related by the equation

$$\frac{dr}{\sqrt{A(r-p)(r-q)(r-\pi)(r-\kappa)}} = \frac{2\sqrt{k}}{\sqrt{A(p-q)(\pi-\kappa)}} \cdot \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}, \tag{19}$$

or

$$\frac{dr}{\sqrt{A(r-p)(r-q)(r-\pi)(r-k)}} = \frac{2\sqrt{k}}{\sqrt{A(p-q)(\pi-\kappa)}} \cdot \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, \tag{20}$$

where

$$\xi = \sin\phi. \tag{21}$$

On making the substitutions for the anchor ring from (15) in (16), (17), (18), and (20), we obtain the following results:

$$r = a + \frac{bP(1-\xi)}{1+\xi-P(1-\xi)}, \tag{22}$$

$$P = -\sqrt{\frac{a}{2(a-b)}}, \tag{23}$$

* Durège, loc. cit., p. 44, ff.

the minus sign being chosen, since it is desired that r and ξ increase simultaneously. Also

$$(24) \qquad k = \frac{\sqrt{2a} - \sqrt{a-b}}{\sqrt{2a} + \sqrt{a-b}},$$

and

$$(25) \quad v = \pm 2\sqrt{\frac{bk}{a+b}} \int_{-1}^{\xi} \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} = \pm 2\sqrt{\frac{bk}{a+b}} \int_{-\frac{\pi}{2}}^{\phi} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}.$$

Let

$$(26) \qquad 2\sqrt{\frac{bk}{a+b}} = \frac{1}{a}.$$

Then in the notation of Durège,* equation (25) may be written

$$(27) \qquad v = \pm \frac{1}{a} \int_{-\frac{\pi}{2}}^{\phi} \frac{d\phi}{\Delta\phi},$$

or

$$(28) \qquad \pm\, av = \int_0^{\phi} \frac{d\phi}{\Delta\phi} + \int_{-\frac{\pi}{2}}^{0} \frac{d\phi}{\Delta\phi} = \int_0^{\phi} \frac{d\phi}{\Delta\phi} + K,$$

or, solving for ϕ,

$$(29) \qquad \phi = \operatorname{am}\,(\pm\, av - K).$$

The formula for calculating $\xi = \sin\phi$ is†

$$\operatorname{sn}(u-v) = \frac{\operatorname{sn}u \operatorname{cn}v \operatorname{dn}v - \operatorname{sn}v \operatorname{cn}u \operatorname{dn}u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v},$$

u, v in this case having the values $\pm\, av$, K. After using the relation $\operatorname{cn} K = 0$ and $\operatorname{sn} K = 1$, this gives

$$(30) \qquad \xi = -\frac{\operatorname{cn}(\pm\, av)\operatorname{dn}(\pm\, av)}{\operatorname{dn}^2(\pm\, av)} = -\frac{\operatorname{cn} av}{\operatorname{dn} av},$$

since $\operatorname{cn} av$ and $\operatorname{dn} av$ are even functions.

It now remains to express u in terms of these elliptic functions. With the help of equation (22), since $\cos u = (a-r)/b$ we obtain

$$(31) \qquad \cos u = \frac{-P(1-\xi)}{1 + \xi + P(1-\xi)}.$$

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* Loc. cit., pp. 8, 17.

† Durège, loc. cit., pp. 106, 116.

Substituting (30) in (31) we find that *the equations defining u in terms of v, which determine the asymptotic lines on the anchor ring, may be expressed in the form:*

$$(32)\quad \begin{cases} (a) & \cos u = -\dfrac{P(\operatorname{dn} av + \operatorname{cn} av)}{\operatorname{dn} av - \operatorname{cn} av - P(\operatorname{dn} av + \operatorname{cn} av)}, \\[2ex] (b) & \sin u = \pm\dfrac{\sqrt{(\operatorname{dn} av - \operatorname{cn} av)^2 - 2P(\operatorname{dn}^2 av - \operatorname{cn}^2 av)}}{\operatorname{dn} av - \operatorname{cn} av - P(\operatorname{dn} av + \operatorname{cn} av)}, \end{cases}$$

where the values of P and a are to be taken from equations (23) *and* (26).

4. The Geometric Character of the Asymptotic Lines. It has already been shown that the asymptotic lines lie on the inner half of the anchor ring. Their form may be determined by examining equations (32) and (9) which give u and du/dv respectively, and also the equation

$$(33)\qquad \frac{d^2u}{dv^2} = \frac{\sin u(2b\cos u - a)}{2b}$$

obtained by taking the v-derivative of (9).

In equation (32, b), the $\pm$ signs indicate that the asymptotic lines are related in pairs symmetrical with respect to the v-axis. Only the positive sign need be considered. The sign of (9) is not necessarily the same as that chosen in (32, b). The correspondence will be evident from the following discussion.

The positive function u defined by equations (32) is, in the first place, periodic in v since the elliptic functions have a period of $4K$, where K is defined by

$$(34)\qquad K = \int_0^{\pi/2} \frac{d\phi}{\Delta\phi}.$$

The period in av is $4K$, that is

$$u\left(v + \frac{4K}{a}\right) = u(v).$$

Therefore to determine the graph of the line, we need only consider u as a function of v in the interval $0 \leqq u \leqq 4K/a$. This interval can be still further restricted since

$$\operatorname{cn}(2K \pm w) = -\operatorname{cn} w, \quad \operatorname{dn}(2K \pm w) = \operatorname{dn} w,$$

and therefore

$$u\left(\frac{2K}{a}+v\right)=u\left(\frac{2K}{a}-v\right).$$

The curve is thus symmetrical with respect to the ordinate $v = 2K/a$, and it follows that its form is completely determined when the function u has been investigated in the interval $0 \leqq u \leqq 2K/a$.

Equations (32) and (9) show that u is a continuous function of v and has a continuous derivative. Moreover, du/dv can vanish only when $\cos u = 0$, that is when $\operatorname{cn} av + \operatorname{dn} av = 0$, and it can be shown that this equation is true in the interval $0 \leqq u \leqq 2K/a$, only for $av = 2K$. The limits between which u varies can be found by putting $v = 0$ and $v = 2K/a$ successively in equations (32). When $av = 0$, since $\operatorname{dn}(0) = 1$ and $\operatorname{cn}(0) = 1$, it follows that $u = 0$. For $av = 2K$, we have $\operatorname{dn} av = 1$, $\operatorname{cn} av = -1$, and therefore $u = \pi/2$. It is evident that for the positive function u defined by equation (32), the sign in equation (9) must be the positive one, and that u constantly increases as v varies from 0 to $2K/a$.

The equation (33) shows that the curve is always concave toward the v-axis in the interval $0 \leqq v \leqq 2K/a$, unless $2b > a$. In any case there is an inflexion point at $u = 0$, and when $2b > a$, there is one also at $u = \cos^{-1}(a/2b)$, at which the curvature changes from convex to concave toward the v-axis.

From the foregoing discussion it is evident that the images in the uv plane of the asymptotic lines have the properties shown in figure 3.

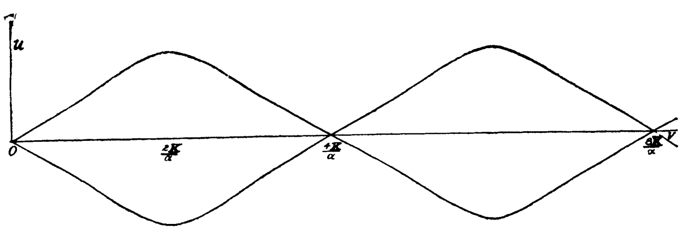

FIG. 3.

On the anchor ring (see figure 4) *the asymptotic lines oscillate back and forth across the inner equator between the two circles $u = \pi/2$ and $u = -\pi/2$. Any two asymptotic lines are similar and can be made to coincide by a rotation about the axis of the ring. A section of one of the lines between*

two successive contacts with the circle $u = -\pi/2$ is symmetrical with respect to a plane through its point of contact with the circle $u = \pi/2$ and the axis of the ring, and any such section may be made to coincide with any other by a rotation about the same axis.

The period of the positive function $u(v)$ defined by equations (32) was found to be $4K/a$. But if the positive values of u in the interval $0 < v < 4K/a$

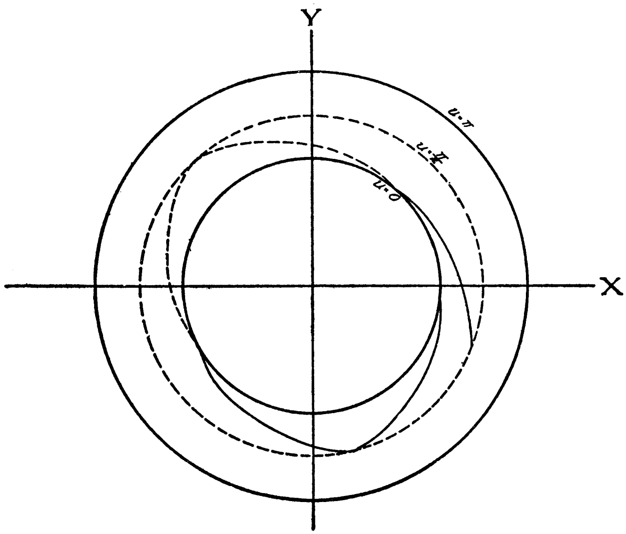

FIG. 4.

are taken with the negative values in $4K/a < v < 8K/a$, and so on as indicated in figure 3, so that the line defined has a continuously turning tangent, the resulting function of v will evidently have the period $8K/a$.

5. Anchor Rings with Closed Asymptotic Lines. The problem here is to determine what form the anchor ring must take if the asymptotic lines are to be closed curves. Since they have a period of $8K/a$, the condition to be satisfied is that some multiple of $8K/a$ must be a divisor of a multiple of 2π, or stating it algebraically,

$$\frac{8Km}{a} = 2\pi n,$$

where m and n are positive integers. From this equation, since K has the value (34), the condition becomes

$$\frac{\pi n}{4m} = \frac{1}{a} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}}. \tag{35}$$

The right hand member depends on a and k and, as is evident from equations (24) and (26), may be considered a function of the ratio b/a. Our first conclusion then is:

The value of the ratio b/a determines whether or not the asymptotic lines are closed; i. e. the closure of the lines depends on the *proportions* and not on the *size* of the ring.

The problem now is to determine whether there is any fixed ratio between a and b which will make the right-hand member of (35) equal to the constant $\pi n/4m$. We will first determine a and k in terms of w where $w = b/a$. Using the values of k and a in (24) and (26). Then

$$k = \frac{\sqrt{2a} - \sqrt{a-b}}{\sqrt{2a} + \sqrt{a-b}} = \frac{\sqrt{2} - \sqrt{1-w}}{\sqrt{2} + \sqrt{1-w}}, \tag{36}$$

$$a = \tfrac{1}{2}\sqrt{\frac{a+b}{bk}} = \tfrac{1}{2}\sqrt{\frac{w+1}{wk}}. \tag{37}$$

If we designate by $\phi(w)$, the function in the right-hand member of (35), the problem is to solve the equation,

$$\phi(w) = \frac{\pi n}{4m}.$$

For this purpose we will study the graph of $\phi(w)$, and since on the anchor ring $a > b$, we may confine ourselves to the interval in which $0 < w < 1$. For $w = 0$, $\phi(w) = 0$, and when $w = 1$, $\phi(w) = \infty$, since $k = 1$ and the integral in (35) is infinite. Therefore the curve passes through the origin and has $w = 1$ as an asymptote.

The derivative $\phi'(w)$ has the value

$$\phi'(w) = -\frac{1}{a^2} \cdot \frac{da}{dw} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} + \frac{1}{a} \int_0^{\pi/2} \frac{k \sin^2\phi \, d\phi}{(\sqrt{1 - k^2 \sin^2\phi})^3} \cdot \frac{dk}{dw}. \tag{38}$$

From (36) we derive

$$(39)\qquad \frac{dk}{dw} = \frac{\sqrt{2}}{\sqrt{1-w}\,(\sqrt{2}+\sqrt{1-w})^2}\,.$$

In (37) a is a function of both w and k; therefore

$$(40)\quad \frac{da}{dw} = \frac{\partial a}{\partial w} + \frac{\partial a}{\partial k}\cdot\frac{dk}{dw}$$

$$= -\frac{1}{4\sqrt{k}}\left[\frac{1}{\sqrt{w^3(1+w)}} + \sqrt{\frac{1+w}{w}}\cdot\frac{\sqrt{2}}{k\sqrt{1-w}\,(\sqrt{2}+\sqrt{1-w})^2}\right].$$

Substituting these values from (39) and (40) in equation (38), it appears that $\phi'(w)$ has the value

$$\frac{1}{4a^2\sqrt{k}}\left[\frac{1}{\sqrt{w^3(1+w)}} + \sqrt{\frac{1+w}{w}}\cdot\frac{\sqrt{2}}{k\sqrt{1-w}(\sqrt{2}+\sqrt{1-w})^2}\right]\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$
$$+\frac{1}{a}\,\frac{\sqrt{2}}{\sqrt{1-w}(\sqrt{2}+\sqrt{1-w})^2}\int_0^{\pi/2}\frac{k\sin^2\phi\,d\phi}{(\sqrt{1-k^2\sin^2\phi})^3},$$

which is positive for $0 < w < 1$, and therefore $\phi(w)$ increases monotonically in that interval.

The general form of the graph of $\phi(w)$ is that of the curve (1) in figure 5. The straight line $\phi = \pi n/4m$ is represented by (2). Curves (1)

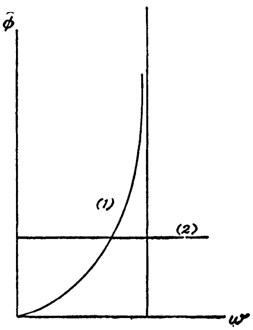

FIG. 5.

and (2) have but one point of intersection, which indicates that *for every value of m and n that may be chosen, there is a fixed ratio between a and b for which*

the asymptotic lines will be closed: i. e., the proportions for a ring may be so chosen that the asymptotic lines close after any arbitrary even number $2m$ of intersections with the inner equator, and an arbitrary number n of windings about the z-axis.

6. A Numerical Example. In this section it will be shown how the branch of the curve given in Figure 3, may be plotted from actual computation. The constants $a = 18$, $b = 14$ have been chosen so as to illustrate the case where there is a point of inflection for $\cos^{-1}(a/2b)$. The values $k = 0.5$ and $\alpha = 4/\sqrt{14}$ are readily computed from equations (24) and (26).

To find K, Landen's first transformation* has been used. The object of this transformation is to increase the modulus k by successive steps, the upper limit of the integral in (34) being decreased incidentally until k is so near to unity that the integral has the approximate value

$$\int_0^{\phi} \frac{d\phi}{\sqrt{1 - \sin^2\phi}} = \log\left[\tan\frac{\phi}{2} + \frac{\pi}{4}\right].$$

The first transformation takes K into

$$K = \frac{2}{1+k}\int_0^{\phi_1} \frac{d\phi_1}{\sqrt{1 - k_1^2 \sin^2\phi_1}},$$

where k_1 and ϕ_1 are determined by the equations

$$k_1 = \frac{2\sqrt{k}}{1+k}, \qquad \sin(2\phi_1 - \phi) = k\sin\phi.$$

After two applications of the formula, k_2 is so near to unity that K may be written in the form

$$K = \sqrt{\frac{k_1}{k}}\,\log\,\tan\left(\frac{\phi_2}{2} + \frac{\pi}{4}\right),$$

where $\frac{\phi_2}{2} + \frac{\pi}{4} = 73° \; 41' \; 1.2''$. Hence we find $K = 1.686$.

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* W. E. Byerly, *Integral Calculus*, p. 215.

The period of the asymptotic lines was found to be $8K/a$, which in the present problem therefore becomes 12.61.

From the equations (32) and (9) of the asymptotic lines, values of u and $\frac{du}{dv}$ corresponding to special values of v may be computed as shown in the following table:

v	u	$\frac{du}{dv}$
0	0	0.524
$\frac{K}{a} = 1.576$	0.927	0.6415
$\frac{2K}{a} = 3.152$	$\pi/2$	0

There is a point of inflection when $u = 0.872$ radians.

To find other points (u, v) on the curve, values of the integral

$$v = \int_0^{\pi/2} \frac{\sqrt{7}}{\sqrt{\cos u(9 - 7\cos u)}}\, du,$$

obtained from equation (13), may be computed by means of the trapezoidal rule for computing areas under a curve.

This rule is: * "Add together the halves of the extreme ordinates and the whole of the intermediate ordinates, and multiply the result by the common interval." To apply the rule, the interval $\pi/2$ was divided into eighteen subdivisions of 0.087 each, the ordinate at each division point calculated and substituted in the formula just given. A table is appended which gives the resulting values of u and v in radians, by means of which the curve in figure 3, may be easily plotted for the interval $0 \leqq v \leqq 2K/a$.

* Williamson, *Integral Calculus*, p. 212.

For the projection of the asymptotic line on the xy plane, shown in

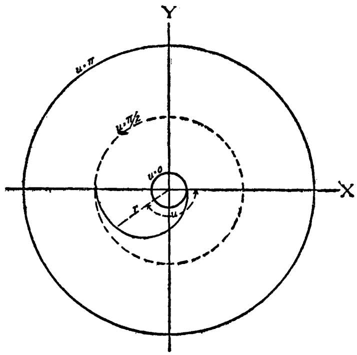

FIG. 6.

figure 6, the values for r were calculated from $r = a - b \cos u$, and these are given in the third column of the table.

TABLE

u	v	r
.000	.000	4.00
.087	.162	4.04
.174	.330	4.20
.262	.481	4.46
.349	.637	4.84
.436	.788	5.30
.524	.934	5.80
.611	1.070	6.30
.698	1.206	7.23
.785	1.350	8.10
.873	1.480	9.10
.960	1.620	9.80
1.047	1.762	11.00
1.134	1.804	12.10
1.228	2.050	13.20
1.309	2.203	14.30
1.396	2.396	15.50
1.483	2.631	16.70
1.570	3.152	18.00

UNIVERSITY OF WISCONSIN, AUGUST, 1906.